On boundary layers in two-dimensional flow with vorticity

By J. F. HARPER

Department of Applied Mathematics and Theoretical Physics,
Free School Lane, Cambridge†

(Received 5 December 1962)

The starting-point for this paper is the suggestion (Batchelor 1956*b*) that the wake behind a bluff body in a uniform stream may consist principally of two eddies rotating in opposite directions. The fluid is assumed to be incompressible and in two-dimensional steady motion at a very high Reynolds number. Along the boundary between the eddies, a viscous layer must form. This layer is unusual in that merely the vorticity, and not the velocity itself, varies appreciably across it. It will be shown that such layers can be treated theoretically much more simply than the general case, because it is possible to linearize the equation of motion. They may, of course, exist in flows other than that past a bluff body.

A discussion is also given of the flow near the rear stagnation point, where this boundary layer meets the body. It had been suggested that a large number of small eddies would have to exist there, but this seems not to be so.

1. Introduction

This paper is concerned with those steady two-dimensional motions of an incompressible fluid which include regions of closed streamlines. We may imagine the limit, as the viscosity tends to zero, of a large class of such flows to be as follows: the motion is inviscid almost everywhere, but is divided by singular streamlines (very near which boundary-layer approximations hold) into a number of regions each of which contains flow which is either irrotational or (see Batchelor 1956*a*) of uniform vorticity.

A particular case, which is of considerable theoretical interest, is Batchelor's (1956*b*) 'closed wake' model for the flow around a bluff body in a uniform stream. Figure 1 illustrates this model for a flat plate set across the stream. Here, according to Batchelor, the dividing streamlines from A and B, the edges of the plate, may meet again at C. Downstream from C there extends a thin laminar wake of the well-known type (Schlichting 1960, Sec. IX f). The region ABC, which has a cusp at C, contains two standing eddies ACD, BCD in each of which the vorticity is uniform. In this particular flow the values of the vorticity in the eddies are equal and opposite, by symmetry.

The dividing streamline (CD) between two eddies may therefore (to an inviscid approximation) be a discontinuity not of the velocity but merely of its normal derivative, which remains finite. The nature of the viscous layer along such a line is the primary subject of the work described here. Because the velocity

† Now at the Department of Mathematics, University of Bristol.

varies only very slightly across the layer, we may linearize the equation of motion in it, and thereby simplify the calculations greatly. This type of simplification was introduced by Moore (1959), who considered the related problem of the boundary layers around gas bubbles in viscous fluids: there too the velocity is uniformly approximated, in the limit of zero viscosity, by that of the inviscid solution, but its normal derivative is not. Moore's calculation was for axisymmetric flow. Proudman (1960) has considered two special cases of two-dimensional boundary layers in which the lowest derivatives of the velocity which differ appreciably from their inviscid values are the third and the fifth; this does not affect the linearization.

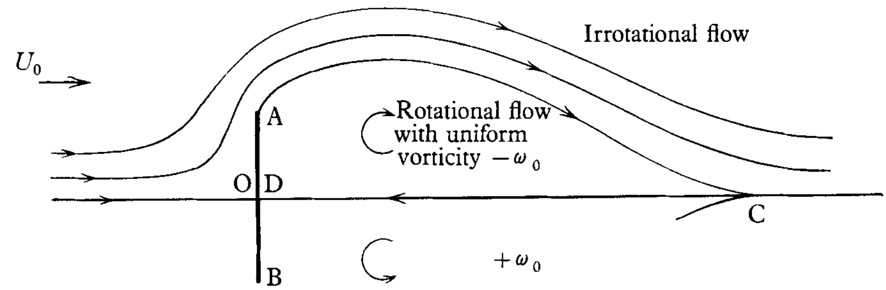

FIGURE 1. Batchelor's proposal for the flow past a two-dimensional flat plate.

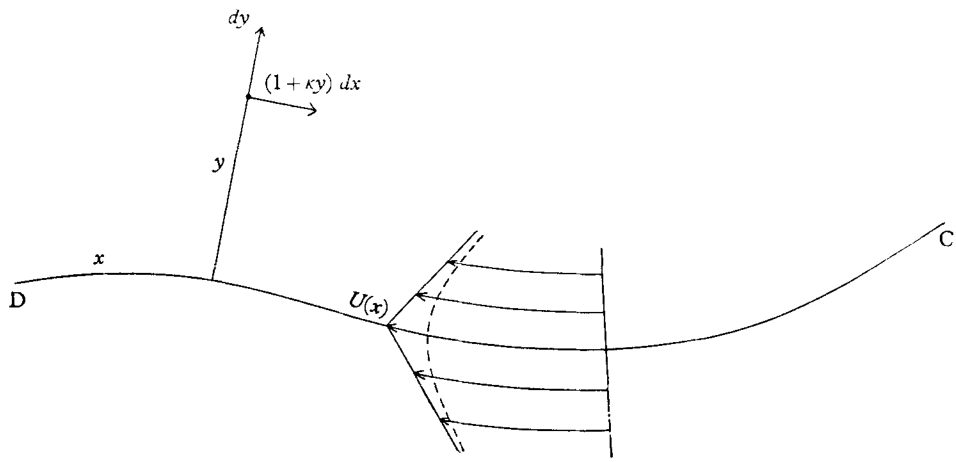

FIGURE 2. The co-ordinate system and inviscid velocity profile near the boundary between eddies. The dotted curve represents the velocity profile in the boundary layer.

It will be noticed that at D in figure 1 there is a stagnation point where our boundary layer meets the surface of the body. Using Fraenkel's (1961) calculation of the leading inviscid term for the flow, we shall discuss the boundary layers near D, both along CD and along AB. The main result is that one need not postulate the existence of a sequence of standing eddies of diminishing size as D is approached; the fluid can turn the corner without them. We remark also that if Fraenkel's solution holds, the vorticity must have equal and opposite values in the two main eddies, even if the flow configuration as a whole is not symmetrical, unless D is at a re-entrant angle of the body.

It should perhaps be made clear that vorticity in the external flow has two entirely different effects on boundary layers. One of these, which has been much studied in recent years, is the second-order effect of this vorticity on a boundary layer whose leading term does not depend on it. The other, which is the subject of this paper, occurs where there would be no viscous layer at all but for the vorticity.

2. The boundary layer with finite vorticity

To study the viscous layer along a dividing streamline between eddies (CD in figure 1), let us use general orthogonal co-ordinates such that x and y denote distances along and perpendicular to the boundary (see figure 2). If κ is the curvature of this line (so that κ is a function of x), the elements of length along the parallel curves and along the normals are $(1+\kappa y)\,dx$ and dy respectively. The co-ordinate system is well defined in a strip extending on each side of the streamline CD sufficiently far for our purposes if κ is everywhere finite. In the important special case of motion symmetrical about a plane, CD is straight (as it is in figure 1), $\kappa \equiv 0$, and the co-ordinate system is Cartesian. Let the inviscid flow be such that the values of the vorticity above and below CD are $(-\omega_1-\omega_0)$ and $(-\omega_1+\omega_0)$ respectively. The velocity $U(x)$ along this line will be assumed to be negative (i.e. in the direction CD).

The stream function ψ may be defined so that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{1+\kappa y}\frac{\partial \psi}{\partial x},$$

and the vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \kappa u.$$

The inviscid approximation to the flow near $y = 0$ is then given by

$$\psi = U(x)\,y - \tfrac{1}{2}U(x)\,\kappa y^2 + \tfrac{1}{2}\omega_0 y\,|y| + \tfrac{1}{2}\omega_1 y^2 + O(y^3), \tag{1}$$

where the first term on the right-hand side represents the stream $U(x)$, the second the correction to that stream which makes it irrotational on $y = 0$, and the remaining terms adjust the vorticity to the prescribed values on each side of $y = 0$.

In the boundary layer near the dividing streamline it is convenient to make the substitutions

$$\left.\begin{aligned} y &= \nu^{\frac{1}{2}}\eta, \\ \psi &= \nu^{\frac{1}{2}}\{U\eta - \tfrac{1}{2}\nu^{\frac{1}{2}}U\kappa\eta^2 + \Psi(x,\eta)\}, \end{aligned}\right\} \tag{2}$$

where ν is the kinematic viscosity of the fluid. The equations of motion in the layer (which are given in full, in the original co-ordinates, by Goldstein 1938, § 45) can then be reduced to

$$U_x\Psi_\eta + U\Psi_{x\eta} - U_x\Psi_{\eta\eta} + \Psi_\eta\Psi_{x\eta} - \Psi_x\Psi_{\eta\eta} = \nu U_{xx} + \Psi_{\eta\eta\eta}, \tag{3}$$

where subscripts denote differentiation. A large number of terms which are easily seen to be negligibly small in the limit as $\nu \to 0$ have been ignored. Equation (1) leads to the boundary condition

$$\Psi \sim \tfrac{1}{2}\omega_0\nu^{\frac{1}{2}}\eta\,|\eta| + \tfrac{1}{2}\omega_1\nu^{\frac{1}{2}}\eta^2 \quad \text{if} \quad T^{\frac{1}{2}} \ll |\eta| \ll \nu^{-\frac{1}{2}}L, \tag{4}$$

where T and L are representative time and length scales of the inviscid motion. Because we are considering the limit of vanishing viscosity we may replace the condition on $|\eta|$ in (4) by the (non-uniformly valid) approximation $|\eta| \to \infty$, as is usual in boundary-layer theories.

In this limiting case it appears from (4) that Ψ may be of order $\nu^{\frac{1}{2}}$ for all finite η. It will be assumed that this is so. The two terms in (3) which are non-linear in Ψ are therefore of order ν and (like νU_{xx}) can be neglected. The consistency of the assumptions will be checked *a posteriori* in the appendix to this paper. The linearized homogeneous form of (3) can be simplified† by the substitutions

$$\left.\begin{aligned} \Psi_\eta &= F(X, Y)/U, \\ \text{where} \qquad X &= \int_{x_0}^{x} U(x')\,dx', \quad Y = U\eta = \nu^{-\frac{1}{2}} U y. \end{aligned}\right\} \tag{5}$$

Here x_0 is a fixed point, to be so chosen that X is positive everywhere along the layer under consideration. We recall that U is negative, so that $x < x_0$. This means that x_0 must be at the upstream end of the layer.

By making the substitutions (5), we obtain

$$F_X = F_{YY} \tag{6}$$

as the simplified equation of motion, and

$$F \to \omega_0 \nu^{\frac{1}{2}} |Y| + \omega_1 \nu^{\frac{1}{2}} Y \tag{7}$$

as the boundary condition for $|Y| \to \infty$. Since $\omega_1 \nu^{\frac{1}{2}} Y$ is an exact solution of (6), we see that only the symmetric part of F is dynamically significant in the boundary layer. The vorticity is given, to a boundary-layer approximation, by

$$-\partial u/\partial y = -\nu^{-\frac{1}{2}} F_Y = -\omega_0 G(X, Y) - \omega_1, \quad \text{say.} \tag{8}$$

G must then satisfy the following equations:

$$G_X = G_{YY}, \tag{9}$$

$$G(X, Y) \to \pm 1 \quad \text{as} \quad Y \to \pm\infty \quad \text{respectively.} \tag{10}$$

Without loss of generality, we may also impose the condition

$$G(0, Y) = \gamma(Y) \tag{11}$$

at the fixed point x_0. For consistency with (10) we require that $\gamma(Y) \to \pm 1$ as $Y \to \pm\infty$.

It follows from the theory of the diffusion equation (see, for example, Carslaw & Jaeger 1947, § 17) that if $\gamma(Y)$ is sufficiently well-behaved—we shall take it to be bounded and integrable in any domain, which is enough—the unique solution to (9), (10), and (11) for positive X is

$$G(X, Y) = \frac{1}{2(\pi X)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \gamma(Y') \exp\{-(Y - Y')^2/4X\}\, dY'. \tag{12}$$

† See von Kármán & Millikan (1934).

We see from this analogy with heat conduction that for $X > 0$ the vorticity is not merely finite but is an analytic function of X and Y.

A particular case of (12) is the similarity solution

$$G_1(X, Y) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^{Y/2X^{\frac{1}{2}}} \exp(-t^2)\, dt = \operatorname{erf}(Y/2X^{\frac{1}{2}}). \tag{13}$$

Here $\gamma(Y) = \pm 1$ according as $Y \gtrless 0$, and $\gamma(0) = 0$. Any solution (12) tends towards the form (13) as X increases, but the inviscid flow may well be such that X has a finite upper bound. Equation (13) therefore need not be a good approximation anywhere. For this similarity solution, the variation of the boundary-layer thickness with X is given by

$$y \propto \frac{\nu^{\frac{1}{2}}}{U} \left(\int_{x_0}^{x} U\, dx \right)^{\frac{1}{2}}. \tag{14}$$

The present theory is therefore unlikely to be valid near zeros of U, i.e. stagnation points. The flow near such points will be discussed in § 3. It will be seen from (14) that, as one would expect, the thickness varies as $(x_0 - x)^{\frac{1}{2}}$ in uniform flow, that the layer broadens more rapidly in decelerating flow, and less rapidly if the flow is accelerated. If the acceleration is sufficient, the layer will actually become thinner.

3. The flow near a stagnation point

At the point D in figure 1, where the dividing streamline meets the rear of the body, there is a stagnation point. The nature of the flow near this point is of interest for several reasons. First, Batchelor (1956*b*) thought it possible that the fluid at the centre of the boundary layer on CD might be brought to rest before reaching D. A secondary pair of eddies would therefore exist there, and perhaps even a whole sequence of such eddies. Secondly, the theory given in § 2 of this paper ceases to hold if U becomes very small, and one would wish to examine this case merely for the sake of completeness. Finally, it has been shown (Fraenkel 1961) that the effects of vorticity dominate the inviscid flow near D, and if the angle between the streamlines which meet there (CD and DA or DB) is a right-angle, the inviscid solution has a logarithmic singularity. In spite of this, there appears to be a remarkably close analogy to the well-known solution (Hiemenz 1911) for the motion near a stagnation point where an irrotational flow divides on reaching a rigid wall. It will also appear that Batchelor's hypothesis of 'secondary eddies' is unnecessary. The essential physical reason for this is that the velocity in the boundary layer on CD is so nearly equal to the velocity just outside it that the fluid in the layer can be brought to rest only by an adverse pressure gradient great enough to do the same to the fluid in the inviscid region.

3.1. *Motion outside the viscous layer on the wall*

Let the angles between the tangents at D to CD and DA, DB be β, β' respectively. We shall consider only the case in which neither β nor β' vanishes, i.e. the streamlines meet at D at finite angles to one another. Use will be made (see figure 3) both of rectangular Cartesian co-ordinates (x, y) with the x-axis along the tangent to DC, and of polar co-ordinates (r, θ) whose initial line is along the bisector of the angle β. In each case the origin is at D. The Cartesian system here defined tends to coincidence, as D is approached, with the orthogonal curvilinear system used in § 2 above. Our calculations will be primarily of the motion in the upper standing eddy (i.e. $y \geqslant 0$); the motion in the lower one could be found by exactly analogous methods.

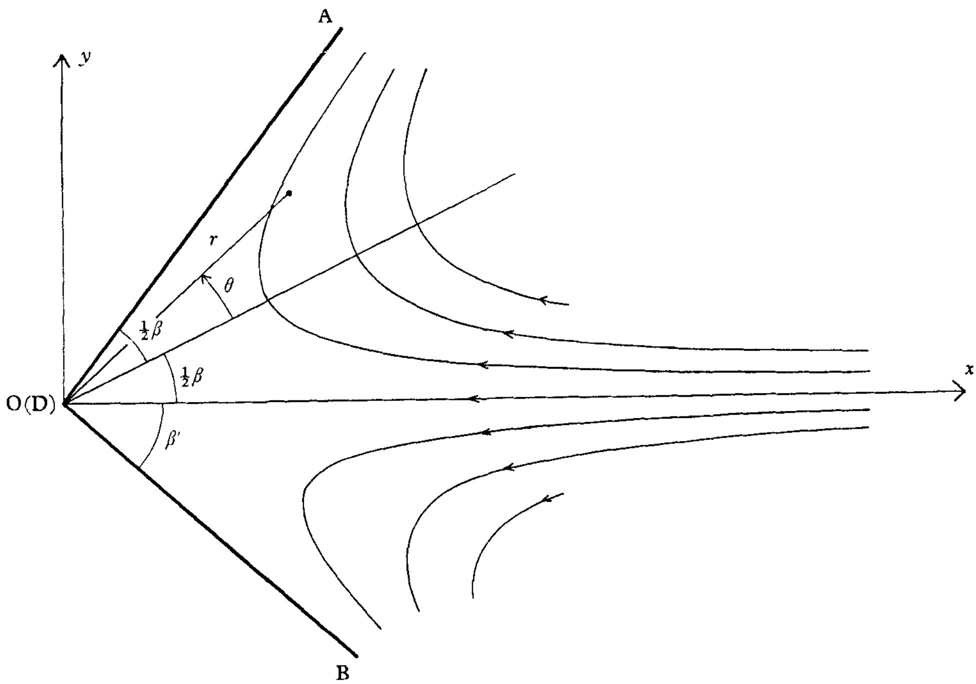

FIGURE 3. The inviscid flow near a stagnation point.

Fraenkel's results for the dominant terms (for sufficiently small r) describing the flow outside the boundary layers are

$$\left.\begin{aligned} \psi_1 &= \tfrac{1}{4}(\omega_0+\omega_1)\, r^2(1-\cos 2\theta/\cos\beta) \quad \text{if} \quad \beta < \tfrac{1}{2}\pi, \\ \psi_2 &= (\omega_0+\omega_1)\,\pi^{-1} r^2\{\tfrac{1}{4}\pi + \ln(r/a)\cos 2\theta - \theta\sin 2\theta\} \quad \text{if} \quad \beta = \tfrac{1}{2}\pi \\ &\text{and} \\ \psi_3 &= -Cr^{\pi/\beta}\cos(\pi\theta/\beta) + \tfrac{1}{4}(\omega_0+\omega_1)\, r^2(1-\cos 2\theta/\cos\beta) \quad \text{if} \quad \pi > \beta > \tfrac{1}{2}\pi, \end{aligned}\right\} \tag{15}$$

where a and C are constants determined by the inviscid motion. They must both be positive. The expressions are valid if the curvature of every dividing streamline is bounded in some neighbourhood of the origin. We assume that this is so.

The values of $U(x)$, as defined in § 2, in the three cases are

$$\left.\begin{aligned} U_1(x) &= -\tfrac{1}{2}(\omega_0+\omega_1)\, x\tan\beta + o(x), \quad \beta < \tfrac{1}{2}\pi, \\ U_2(x) &= (\omega_0+\omega_1)\,\pi^{-1}x\{2\ln(x/a)+1\} + o(x), \quad \beta = \tfrac{1}{2}\pi, \\ U_3(x) &= -(\pi/\beta)\, Cx^{(\pi/\beta)-1} - \tfrac{1}{2}(\omega_0+\omega_1)\, x\tan\beta + o(x), \quad \beta > \tfrac{1}{2}\pi. \end{aligned}\right\} \tag{16}$$

Since $(\omega_0+\omega_1)$ and C are positive, the U_i are negative for sufficiently small x. The analogous results calculated from the lower eddy are found by replacing $(\omega_0+\omega_1)$ and β by $(\omega_0-\omega_1)$ and β' respectively. Because Bernoulli's theorem must hold for the inviscid motion, and the origin is a stagnation point for both eddies, the values of U_i must be equal. It follows immediately that $\beta=\beta'$ and $\omega_1=0$, unless both β and β' are less than $\frac{1}{2}\pi$, in which case we have only that

$$(\omega_0+\omega_1)\tan\beta = (\omega_0-\omega_1)\tan\beta'.$$

This means that the two eddies in the wake of a body without re-entrant cavities must have equal and opposite vorticities, because $\beta+\beta' \geqslant \pi$ for such a body, wherever the rear stagnation point happens to be. Because $\beta=\beta'$, the motion is symmetrical about Ox, as far as its leading terms near O are concerned. For simplicity, we shall assume that $\beta=\beta'$ and $\omega_1=0$ even if $\beta<\frac{1}{2}\pi$.

The boundary-layer theory in § 2 can be applied if the calculated perturbations to $u=U_i(x)$ are very small by comparison with $U_i(x)$. A fuller discussion of the validity of the approximations will be deferred to the appendix, in the interests of clarity here. The result is that if $U > O(\nu^{\frac{1}{4}})$, which is in all cases true if $x > O(\nu^{\frac{1}{4}})$, the boundary-layer theory is still valid (or, strictly speaking, self-consistent: it has not been proved that the solution *must* be of this form anywhere).

If x is very small, however, the layer is thickening rapidly and vorticity gradients are decreasing, which suggests that the viscous terms may become unimportant for sufficiently small x. It will now be shown that for this to happen it suffices that $x=o(1)$, so that there is a region $O(\nu^{\frac{1}{4}}) < x = o(1)$ where the layer is effectively inviscid. We do so by comparing the orders of magnitude of the various terms in the vorticity equation, a boundary-layer approximation to which is

$$\psi_y\omega_x-\psi_x\omega_y = \nu\omega_{yy},$$

where ω is the local vorticity. If our linear theory holds,

$$\begin{aligned}\psi_y\omega_x &\sim \omega_0(U^2G_X+\nu^{-\frac{1}{2}}yUU_xG_Y),\\ \psi_x\omega_y &\sim \omega_0\nu^{-\frac{1}{2}}yUU_xG_Y,\\ \nu\omega_{yy} &\sim \omega_0U^2G_{YY}.\end{aligned}$$

(We recall that $G_X=G_{YY}$.) The ratio of inertial to viscous terms is then

$$\begin{aligned}\mathrm{I/V} &= \omega_0\nu^{-\frac{1}{2}}yUU_xG_Y/\omega_0U^2G_{YY}\\ &= (YG_Y/XG_X)(XU_x/U^2). \qquad (17)\end{aligned}$$

Both of these factors in parentheses are dimensionless. The first (YG_Y/XG_X) depends only on the solution of the diffusion equation given in § 2. If we make the assumption that the vorticity is a monotonic function of Y at $X=0$ (the upstream end of the layer), which seems intuitively likely, YG_Y/XG_X is finite and non-zero at all points where X and Y are. In particular, for the similarity solution (13) its value is -2 everywhere. The second factor in (17), however, tends to infinity as $x\to 0$, because $U\to 0$ and neither X nor U_x does. (In fact $U_x\to\infty$ if $\beta\geqslant\frac{1}{2}\pi$.) XU_x/U^2 cannot tend to infinity at any other point on the boundary except the upstream end, if there is a suitable flow régime there. But the upstream end is outside the scope of the present theory in any case. We note in passing that XU_x/U^2 is finite and constant if either $X \propto (x_0 - x)^n$ or $X \propto e^{(x_0-x)/b}$, where n and b are constants.

It can be concluded that viscosity is negligible in the boundary layer at points sufficiently near O for XU_x/U^2 to be very large if the boundary-layer assumptions are still valid. The condition is, as mentioned above, that $O(\nu^{\frac{1}{2}}) < x = O(1)$. The distribution of vorticity in this region can now be found from (8), with ω_1 put equal to zero. We may replace X with negligible error here by

$$\int_{x_0}^{0} U(x')\,dx' = c^2, \quad \text{say.}$$

Also, since the perturbations to the stream $U(x)$ are small in the boundary layer, a first approximation to the stream function ψ is $\nu^{\frac{1}{2}} Y$. The vorticity is then, in this approximation, $-\omega_0 G(c^2, \nu^{-\frac{1}{2}}\psi)$, which is a function of ψ, as required for an inviscid flow.

We now assume that the motion remains inviscid, with this dependence of vorticity on ψ, for $x = O(\nu^{\frac{1}{2}})$, where the flow is by no means unidirectional. The suggested equation governing the motion is then

$$\nabla^2\psi = \omega_0 G(c^2, \nu^{-\frac{1}{2}}\psi). \tag{18}$$

The possibility of a viscous boundary layer petering out into an inviscid flow, with only the distribution of vorticity left to remind us of its viscous origin, is not mentioned in any other work on the subject known to me.† It may well occur in other circumstances where a boundary layer approaches a point at which Prandtl's equations for the motion in the layer break down.

Equation (18) is intractably non-linear, and no general methods of solution (apart from term-by-term evaluation of series, and numerical computation) appear to be known. We can say, however, that its solutions need be appreciably different from those of Fraenkel for the exterior flow only where $\psi = O(c\nu^{\frac{1}{2}})$. The 'inviscid boundary layer' therefore remains thin. In the region where $\psi \ll c\nu^{\frac{1}{2}}$, which is the immediate neighbourhood of the dividing streamlines through O, (18) can be written as

$$\nabla^2\psi \doteqdot \omega_0 \nu^{-\frac{1}{2}} G'\psi = \lambda^{-2}\psi, \tag{19}$$

where $G' = [\partial G(c^2, Y)/\partial Y]_{Y=0}$ and λ is a constant with the dimensions of a length and of order of magnitude $\nu^{\frac{1}{4}}$. The solution of (19) which holds near the origin is

$$\psi = A(r/\lambda)^{\pi/\beta}\cos(\pi\theta/\beta) + O\{(r/\lambda)^{2\pi/\beta}\}, \tag{20}$$

where A is a constant, because $\psi = 0$ where $\theta = \pm\frac{1}{2}\beta$, and there is no singularity at the origin. It seems unlikely that A could vanish, for if it did there would have to be more dividing streamlines reaching the stagnation point in the arc $-\frac{1}{2}\beta < \theta < \frac{1}{2}\beta$.

We now have approximations to the solution near the origin both for $\psi \gg c\nu^{\frac{1}{2}}$ and for $\psi \ll c\nu^{\frac{1}{2}}$. If $\beta \geqslant \frac{1}{2}\pi$ it is also possible to find one for $\psi = O(c\nu^{\frac{1}{2}})$ without

† Except that of Moore (1963), which I first saw after the present work had been submitted for publication. He has found the same phenomenon in axisymmetric flow while continuing his study (see Moore 1959) of the flow around a spherical bubble.

having to solve (18). The method for doing this is most easily seen for the case $\beta > \frac{1}{2}\pi$, which will be dealt with first. We shall then consider the case $\beta = \frac{1}{2}\pi$.

Let us assume that if $\beta > \frac{1}{2}\pi$, $\psi \sim \psi_3$ (defined as in equation (5)) for large values of $\psi/c\nu^{\frac{1}{2}}$, or in other words that the 'inviscid boundary layer' matches to Fraenkel's solution for the flow outside it. We introduce dimensionless scaled co-ordinates by putting

$$\psi = c\nu^{\frac{1}{2}}\phi_3, \quad r = (c\nu^{\frac{1}{2}}C^{-1})^{\beta/\pi} r_3, \tag{21}$$

so that $\psi_3 = O(c\nu^{\frac{1}{2}})$ where $r_3 = O(1)$. Equation (18) becomes in these co-ordinates, with the obvious definition of ∇_3^2,

$$\begin{aligned} \nabla_3^2 \phi_3 &= (c\nu^{\frac{1}{2}})^{2\beta/\pi-1} C^{-2\beta/\pi}\omega_0 G(c^2, c\phi_3) \\ &= k_3 G(c^2, c\phi_3) = k_3 g(\phi_3), \quad \text{say.} \end{aligned} \tag{22}$$

The assumption that $c\nu^{\frac{1}{2}} \phi_3 \sim \psi_3$ for $r_3 \gg 1$ leads to

$$\phi_3 \sim -r_3^{\pi/\beta}\cos(\pi\theta/\beta) + \tfrac{1}{4}k_3 r_3^2(1-\cos 2\theta/\cos\beta), \tag{23}$$

and we also have the boundary condition that $\phi_3 = 0$ on the dividing streamlines $\theta = \pm\frac{1}{2}\beta$.

Since $g(\phi)$ is a bounded function and $k_3 \propto \nu^{\beta/\pi - \frac{1}{2}}$, which tends to zero as ν does, the forms of (22) and (23) suggest that ϕ_3 can be written as

$$\phi_3 = \Phi_{31}(r_3, \theta) + k_3\, \Phi_{32}(r_3, \theta) + o(k_3), \tag{24}$$

where Φ_{31} and Φ_{32} are independent of ν (or k_3). If this is so, we see from (22) that Φ_{31} must be a harmonic function and then from (23) that

$$\Phi_{31} = -r_3^{\pi/\beta}\cos(\pi\theta/\beta), \tag{25}$$

for any other form would either have a singularity in the flow field or not obey the boundary conditions. Replacement of $c\phi_3$ by its dominant term in (22) then yields the following Poisson equation for the lesser term

$$\nabla_3^2\, \Phi_{32} = g(\Phi_{31}). \tag{26}$$

The replacement is valid if $k_3\,\Phi_{32} \ll \Phi_{31}$, which can be seen from (20) to be true near $r_3 = 0$, where, in fact, $\Phi_{32} = O(\Phi_{31})$, and from (23) to be true for $r' \leqslant r_3 \leqslant r''$, where r' and r'' are constants such that $r' \gg 1$ and r'' can be chosen to be arbitrarily large if k_3 is sufficiently small. It seems reasonable to suppose that the solution of (26) is such as to make it valid where $r_3 = O(1)$; all that is needed for this is that Φ_{32} be finite there.

I have not been able to solve (26), even when G has the similarity form (13) and $g(\Phi) = \operatorname{erf}(\frac{1}{2}\Phi)$. The exact nature of Φ_{32} is, however, of less importance than the fact that the irrotational term Φ_{31} remains dominant in the inviscid boundary layer, having become so in the inner parts of the outer region where Fraenkel's solution ψ_3 holds (provided, of course, that the various hypotheses about the behaviour of the flow are correct). It is hardly a matter for surprise that (26) is not uniformly valid for $r_3 \to \infty$; most boundary-layer theories fail in that way at their outer limits.

For the case $\beta = \frac{1}{2}\pi$, dimensionless scaled co-ordinates are defined, for reasons similar to those used above, by

$$\psi = c\nu^{\frac{1}{2}}\phi_2, \quad r = c^{\frac{1}{2}}\omega_0^{-\frac{1}{2}}\nu^{\frac{1}{4}}\{\ln(c^2/\nu)\}^{-\frac{1}{4}} r_2, \tag{27}$$

so that now $\psi_2 = O(\nu^{\frac{1}{2}})$ where $r_2 = O(1)$. Equation (18) becomes

$$\nabla_2^2 \phi_2 = \{\ln(c^2/\nu)\}^{-1} g(\phi_2) = k_2 g(\phi_2), \quad \text{say.} \tag{28}$$

Like k_3 in the analogous equation (22), $k_2 \to 0$ as $\nu \to 0$. The condition analogous to (23) is now

$$\phi_2 \sim c^{-1}\nu^{-\frac{1}{2}}\psi_2 = \pi^{-1}\{-\tfrac{1}{4} + \tfrac{1}{2}k_2 \ln k_2\} r_2^2 \cos 2\theta$$
$$+ k_2 \pi^{-1} r_2^2\{\tfrac{1}{4}\pi + \ln(c\omega_0^{-\frac{1}{2}}a^{-1})\cos 2\theta + \ln r_2 \cos 2\theta - \theta \sin 2\theta\}, \tag{29}$$

for large values of r. It will be seen that ϕ_2 is also dominated by an irrotational term if $\ln r_2 \ll k_2^{-1}$, and reasoning analogous to that given above leads to the hypothesis that

$$\phi_2 = \Phi_{21}(r_2, \theta) + k_2 \ln k_2\, \Phi_{22}(r_2, \theta) + k_2\, \Phi_{23}(r_2, \theta) + o(k_2), \tag{30}$$

and to the conclusions that

$$\Phi_{21} = -\tfrac{1}{2}\Phi_{22} = -(4\pi)^{-1} r_2^2 \cos 2\theta = -(2\pi)^{-1} x_2 y_2, \tag{31}$$

$$\nabla_2^2\, \Phi_{23} = g(\Phi_{21}). \tag{32}$$

Although the actual flow near the stagnation point cannot now be a small perturbation of ψ_2, because there is no term in $\ln r$ in (20), it appears from (30) and (31) that it is only slightly perturbed from that irrotational flow which is a first approximation to ψ_2 where r_2 is of finite order (more exactly $|\ln r_2| \ll k_2^{-1}$), i.e. where ψ_2 itself is of order $c\nu^{\frac{1}{2}}$. The rate of strain is very large in the irrotational stagnation flow at the corner, because of the logarithmic singularity of ψ_2; by transforming back to the original co-ordinates this rate of strain $(-\alpha)$ is easily seen to be

$$-\alpha = -(2\pi)^{-1}\omega_0\{\ln(c^2/\nu) + 2\ln\ln(c^2/\nu) + O(1)\}. \tag{33}$$

If the stagnation point under consideration is in the body of the fluid, it seems possible that viscous boundary layers may form again where the flow converges after leaving the neighbourhood of the stagnation point. Because of the symmetry, the value of X will be given initially by c^2. Such flows could exist (as far as our approximation to the local mechanics is concerned) for $\beta = \pi/N$, where N is any integer $\geqslant 2$, with vorticity alternately $\pm\,\omega_0$ in the sectors.

3.2. *Motion in the viscous layer on the wall*

If the stagnation point which we have been considering is on an impermeable rigid boundary to the fluid, the boundary layer on the wall sufficiently near that point must have one of the similarity forms calculated by Hiemenz (1911) for the case $\beta = \frac{1}{2}\pi$, and by Falkner & Skan (1930) for $\beta \neq \frac{1}{2}\pi$. This is a consequence of (20).

Because there is a logarithmic term in the second of equations (15), the structure of the layer which will form on the wall if $\beta = \frac{1}{2}\pi$ requires closer examination;

it is not obvious *a priori* that there exists a similarity form for the layer which will hold in spite of the logarithmic singularity.

At points sufficiently near O, the inviscid flow is given approximately by the stream function $\Phi_{21} + k_2 \ln k_2 \Phi_{22}$ of § 3.1,

$$\psi = -\alpha xy, \quad v = -\partial\psi/\partial x = \alpha y, \tag{34}$$

α being defined as in (33).

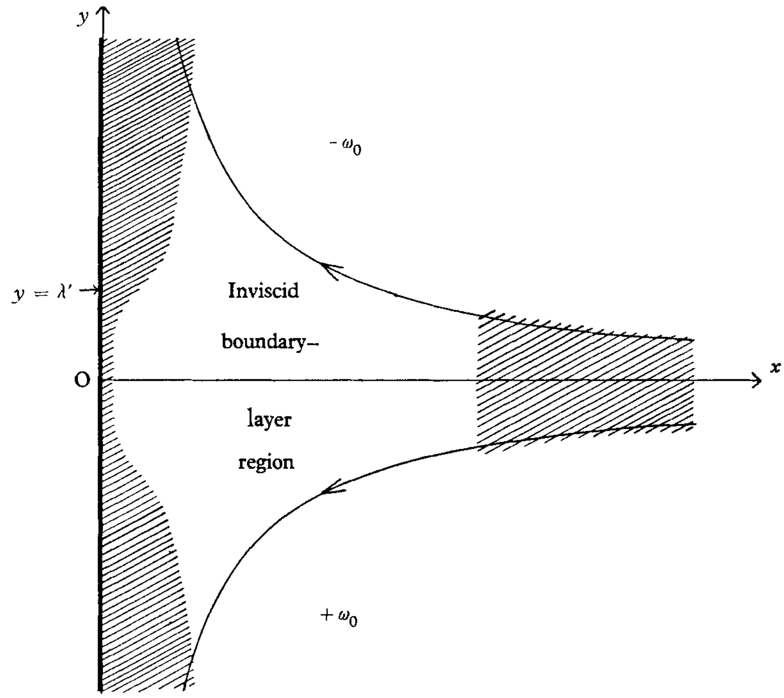

FIGURE 4. The flow near a stagnation point on a rigid wall, as suggested in § 3.2. The shaded regions indicate the extent of the viscous boundary layers.

(We are here considering the boundary layer on Oy; see figure 4.) This approximation, which is an irrotational stagnation flow towards the wall, with uniform rate of strain $-\alpha$, holds if r_2 is of order unity, i.e. $r = O(\lambda' k_3^{\frac{1}{2}})$, where $\lambda' = c^{\frac{1}{4}}\omega_0^{-\frac{1}{2}}\nu^{\frac{1}{4}}$. The boundary layer on the wall is accordingly of the Blasius–Hiemenz similarity form

$$\psi = -(\alpha\nu)^{\frac{1}{2}} y f(\xi),$$

where

$$\xi = (\alpha/\nu)^{\frac{1}{2}} x,$$

and $f(\xi)$ satisfies the differential equation

$$f'^2 - ff'' = 1 + f''', \tag{35}$$

and the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{36}$$

The properties of these equations are well known and need not be given here; the details can be found in Schlichting (1960), Sec.v.10.

If $y \gg \lambda'$, it can be seen from the symmetry of the inviscid flow about the line $y = x$ that the tangential velocity just outside the layer on the wall is

$$V = 2\omega_0 \pi^{-1} y \ln(a/y) + O(y). \tag{37}$$

Suppose also that $y \ll a$. The dominant term in this expression for V is then the first. Let us put

$$\psi = -\{2\nu\omega_0 \pi^{-1} \ln(a/y)\}^{\frac{1}{2}} y f_1(\xi_1) + h(y, \xi_1), \tag{38}$$

where

$$\xi_1 = \{2\nu^{-1}\omega_0 \pi^{-1} \ln(a/y)\}^{\frac{1}{2}} x. \tag{39}$$

We find that $f_1(\xi_1)$ must satisfy exactly the same differential equation (35) and boundary conditions (36) as $f(\xi)$. The error made by neglecting the second term, $h(y, \xi_1)$, in the expression for ψ can be shown to be much less than the leading term, because $\ln(a/y)$ is large. (I am indebted to Dr I. Proudman for suggesting that this approximate similarity solution might exist.)

This result means that the Blasius–Hiemenz similarity solution can still be used (as modified) where $a/\lambda' \gg y/\lambda' \gg 1$, as well as in the original range $y/\lambda' \ll k^{\frac{1}{3}} \ll 1$. It seems reasonable to expect that a similar artifice would enable us to continue this type of solution through the range near $y = \lambda'$, in which V changes smoothly from the form (34) to the form (37), if the exact manner of variation of V were calculated in that range. It also seems likely that this type of motion can exist only if the general flow is towards the wall, as is known to be the case when the motion outside the boundary layer is irrotational everywhere.

Figure 4 contains a sketch of the regions in which viscous boundary layers exist, as suggested in this section. It appears that the layer on the wall can adapt itself to the inviscid flow field without secondary eddies near the stagnation point, even though that field has the rather unusual structure described in § 3.1.

I am grateful to Dr I. Proudman for much helpful discussion and criticism, and to the University of New Zealand for a postgraduate scholarship.

Appendix

We now justify the basic assumption about the smallness of perturbations which was made in §§ 2 and 3.1.

Equation (3) is reducible to the form (6) if the following three conditions hold:

$$|\Psi_\eta| \ll |U|, \tag{A 1}$$

$$|\Psi_x \Psi_{\eta\eta}| \ll |\Psi_{\eta\eta\eta}|, \tag{A 2}$$

$$|\Psi_{\eta\eta\eta}| \gg \nu |U_{xx}|. \tag{A 3}$$

For the solutions of (6), these conditions are:

$$|F| \ll U^2,$$

$$\nu^{\frac{1}{2}}\omega_0 G^2/U^2 \ll |G_Y|,$$

$$|G_Y| \gg |\nu^{\frac{1}{2}} U_{xx}/\omega_0 U|.$$

In order to gain a physical interpretation of these results without undue algebraic complexity, we shall give their explicit forms for the similarity solution (13). The conditions can be written, respectively, as

$$|U/\omega_0| \gg 2^{\frac{1}{2}}\pi^{-\frac{1}{4}}c^{\frac{1}{2}}\omega_0^{-\frac{1}{2}}\nu^{\frac{1}{4}} = 2^{\frac{1}{2}}\pi^{-\frac{1}{4}}\lambda', \tag{A 4}$$

$$|y/\nu^{\frac{1}{2}}| \ll \{-2U^{-2}X\ln(\pi\nu\omega_0^2X/U^4)\}^{\frac{1}{2}}, \tag{A 5}$$

$$|y/\nu^{\frac{1}{2}}| \ll \{-2U^{-2}X\ln(\pi\nu U_{xx}^2X/4\omega_0^2)\}^{\frac{1}{2}}. \tag{A 6}$$

Equation (A 4) is seen to give the order of magnitude of U down to which the theory still holds, and (A 5) and (A 6) then give the upper limits on $|y|$. Although it is always possible that $|y/\nu^{\frac{1}{2}}| > O(1)$ in the limit as $\nu \to 0$, the validity is non-uniform. This is a general property of boundary-layer theories, which was only to be expected here. A less stringent form of (A 5), based on $|U_x\eta| \gg |\Psi_x|$ instead of (A 2), can be given if $U_x \neq 0$, but there is little point in doing so, in view of (A 6). The range of validity permitted by (A 6) is considerably smaller than those one finds in ordinary boundary-layer theory; the reason is that we are requiring a perturbation quantity to be large when compared with one derived (albeit with a factor ν) from the inviscid flow.

Viscosity is negligible in the boundary layer along the dividing streamline if XU_x/U^2 is very large. If $U^2/|F|$ is also very large, both the inviscid and the boundary-layer approximations are valid. It is obviously possible for both conditions to be satisfied if ν is sufficiently small, because the requirement is that U be larger than $O(\nu^{\frac{1}{4}})$ but smaller than $O(1)$. The central hypothesis of the present work is that the inviscid approximation continues to hold for $U = O(\nu^{\frac{1}{4}})$, where the boundary-layer one does not. This hypothesis can be tested by the consistency of the asymptotic solutions obtained from it. No reason is apparent from the calculations in § 3 why it should not be valid.

REFERENCES

BATCHELOR, G. K. 1956*a* *J. Fluid Mech.* **1**, 177.

BATCHELOR, G. K. 1956*b* *J. Fluid Mech.* **1**, 388.

CARSLAW, H. S. & JAEGER, J. C. 1947 *Conduction of Heat in Solids.* Oxford University Press.

FALKNER, V. M. & SKAN, S. W. 1930 *Aero. Res. Counc., Lond., Rep. and Mem.* no. 1314.

FRAENKEL, L. E. 1961 *J. Fluid Mech.* **11**, 400.

GOLDSTEIN, S. (ed.) 1938 *Modern Developments in Fluid Dynamics*, vol. I. Oxford University Press.

HIEMENZ, K. 1911 *Dingl. Polytech. J.* **326**, 321.

KÁRMÁN, TH. VON & MILLIKAN, C. B. 1934 *N.A.C.A. Rep.* no. 504.

MOORE, D. W. 1959 *J. Fluid Mech.* **6**, 113.

MOORE, D. W. 1963 *J. Fluid Mech.* **16**, 161.

PROUDMAN, I. 1960 *J. Fluid Mech.* **9**, 593.

SCHLICHTING, H. 1960 *Boundary Layer Theory*, 4th ed. New York: McGraw-Hill.